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2009 J. Phys. A: Math. Theor. 42 075306

(http://iopscience.iop.org/1751-8121/42/7/075306)

J. Phys. A: Math. Theor. **42** (2009) 075306 (22pp) doi:10.1088/1751-8113/42/7/075306

Thermal and squeezing effects in self-similar potential systems

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Received 2 October 2008, in final form 10 December 2009
Published 21 January 2009
Online at stacks.iop.org/JPhysA/42/075306

Abstract
We discuss the squeezing effects for quadrature observables considering self-similar potential systems which are described either by a purely squeezed state or by a superposition of purely coherent states. We also discuss the statistical properties of these systems via thermal states considering either a purely coherent state or a purely squeezed state representation. The thermal expectations for a quantum canonical ideal gas of such potential systems are also calculated in the two representations and compared.

PACS numbers: 03.65.Ca, 03.65.Fd

1. Introduction

The factorization method [1] is used with success in the study of important eigenvalue problems in physics. The Darboux transformation technique for linear differential equations of second order [2] is the simplest differential operator realization of this method. Schrödinger was responsible for the development of the method in this particular case [3]. Because of its connection with quantum mechanics and supersymmetry theory, the Schrödinger method became well known among physicists. The concept of supersymmetry was first introduced in the early 1970s [4] in the context of a unifying treatment of bosonic and fermionic parts of the string spectrum and has become today a field in its own right with many applications to several fields of physics. In supersymmetric quantum mechanics [5] one-dimensional partner Hamiltonians $\hat{H}_\pm$ are written in terms of the operator $\hat{A}(a_1) \equiv W(a_1; x) + \mathrm{i}\hat{p}_x$ and its adjoint operator as

$$\hat{H}_- = \hat{p}_x^2 + V^{(-)}(a_1; x) = \hat{A}^\dagger(a_1)\hat{A}(a_1) \qquad \text{and} \qquad \hat{H}_+ = \hat{p}_x^2 + V^{(+)}(a_1; x) = \hat{A}(a_1)\hat{A}^\dagger(a_1), \tag{1}$$

where a_1 is a set of potential parameters and the superpotential $W(a_1; x)$ is a real function related to the partner potentials via

$$V^{(\pm)}(a_1; x) = W^2(a_1; x) \pm W'(a_1; x), \tag{2}$$

where dash denotes the derivative with respect to x. For simplicity in this paper we use the system of units where $\hbar = 2m = 1$. For a large class of potential systems, such pairs of Hamiltonians $\hat{H}_\pm$ have a property of reparametrization invariance manifested by the integrability condition $\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1)$, called *shape invariance* [6], where a_2 is a function of a_1 (say $a_2 = f(a_1)$) and the remainder $R(a_1)$ is independent of the dynamical variables. Although not all exactly solvable problems are shape-invariant [7], supersymmetric quantum mechanics together with the shape invariance concept is a powerful and elegant algebraic technique to study exactly solvable systems [8, 9]. Supersymmetric quantum mechanics also found many applications in physics [10–15]. In the case of well-known analytically solvable potentials found in quantum mechanics texts, the sets of parameters a_1 and a_2 are related by a translation [5, 8, 16]. One new class of shape-invariant potentials which are reflectionless and have an infinite number of bound states was introduced by Shabat and Spiridonov [17, 18]. They analyzed an infinite chain of Hamiltonians $\hat{H}_n = \hat{p}_x^2 + V_n(x)$, with $n = 0, 1, 2, \ldots$, the potentials of which are related by $V_{n+1}(x) = V_n(x) + 2W_n'(x)$ with $V_0(x) = W_0^2(x) - W_0'(x) + R_0$. In this case, the infinite chain of differential equations relating the various superpotentials $W_n(x)$ and the constants R_n

$$W_n^2(x) + W_n'(x) = W_{n+1}^2(x) - W_{n+1}'(x) + R_{n+1}, \qquad n = 0, 1, 2, \ldots \tag{3}$$

also called *the dressing chain* in the theory of solitons, reduces to only one equation if we adopt the *ansatz*

$$W_n(x) = W(a_n; x), \qquad a_n \equiv \underbrace{f(f(\cdots f}_{n \text{ times}}(a_0)\cdots)), \tag{4}$$

where a_n is a parameter generated from a_0 by a function f. In fact, with (4) into (3) it follows that the relation

$$W^2(a; x) + W'(a; x) = W^2(f(a); x) - W'(f(a); x) + R(f(a)) \tag{5}$$

is identically satisfied with respect to a, and the set of equations (3) does not depend on n. In terms of the operators $\hat{A}(a)$ and $\hat{A}(f(a))$, the relation (5) can be written as the shape-invariant condition $\hat{A}(a)\hat{A}^\dagger(a) = \hat{A}^\dagger(f(a))\hat{A}(f(a)) + R(f(a))$. When it is assumed that $f(u) = qu$, then the set of potential parameters is related by a scaling: $a_n = q^n a_0$, with the range of the *scaling parameter* values given by $0 < q < 1$. The shape-invariant *self-similar potentials* discovered by Shabat and Spiridonov [17, 18] are obtained if we demand that all superpotentials $W_n(x)$ satisfy the *ansatz*: $W(a_n; x) = q^n W(q^n x)$. A self-similar potential can be considered as quantum deformation of the single-soliton solution corresponding to the Rosen–Morse potential [18]. Indeed, working with this kind of potential it is possible to obtain as limiting cases some important potentials utilized to model quantum confined systems [9], such as the Rosen–Morse, harmonic oscillator and Pöschl–Teller potentials. Relevant applications of the self-similar potential formalism in physics are related to statistical mechanics [19, 20]. As examples of these applications we can cite the infinite soliton solutions of the Korteweg–de Vries equation [21], Ising model for a spin lattice gas in an external magnetic field [22], random matrices [21], lattice gas model on a line for a two-dimensional Coulomb gas [23], etc. Other applications of the self-similar potential come from its connection with the harmonic oscillator quantum deformation algebras, which permits us to establish a relation between the self-similar potential system with the presence of anharmonic contributions in the ordinary harmonic oscillator potential [24] or, as a second possibility, to associate the self-similar potential with the quantum deformed interacting boson models [25]. These models used to be applied in the study of thermodynamics properties of molecules and solids and, in this case, the anharmonic contributions in the thermodynamics quantities are described by quantum deformed bosons.

Although all the applications of the self-similar potentials mentioned above are related to statistical studies of systems, the second set of applications is the inspiration for the present study. In this paper, within a self-consistent algebraic framework, we investigate some thermal and squeezing properties for self-similar potential systems via two different quantum state representations: one when the quantum state of the system is associated with a purely coherent state and another when it happens through a purely squeezed state. As basic tools of this study we use the general algebraic formalism for the shape-invariant systems presented in [8] and its capacity to build generalized special quantum states for these systems, such as purely coherent, purely squeezed or intelligent states [26]. In this sense, this paper is organized as follows. In section 2 we briefly recall the algebraic formulation to shape invariance and the construction process of purely coherent and purely squeezed states for self-similar potential systems. In section 3, we study and compare the squeezing when the self-similar potential system is described by a purely squeezed state and by a superposition of purely coherent states. In section 4 we carry out a detailed discussion on the statistical properties of a gas of self-similar potential systems using thermal states constructed via: (i) purely coherent and; (ii) purely squeezed states. Brief remarks close the paper in section 5.

2. A short review of the algebraic formulation to self-similar potential systems

2.1. Algebraic model

Recently, the use of operator techniques based on algebraic models [8, 9, 27, 28] brought renewed interest to the study of shape-invariant systems. In [8], one introduces the parameter translation operator for the self-similar potential systems [29, 30],

$$\hat{T} \equiv \exp\left\{(\log q) a_1 \frac{\partial}{\partial a_1}\right\}, \tag{6}$$

which acts in the a_n-potential parameters space $\mathcal{E}_a \equiv \{|a_n\rangle; n = 1, 2, 3, \ldots\}$ through the similarity transformation $\hat{T} O(a_1) \hat{T}^\dagger = O(a_2)$. Thus, with the definition of the operators $\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}$ and $\hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger \hat{A}(a_1)$, the partner Hamiltonians of equation (1) can be written in the forms $\hat{H}_- = \hat{\mathcal{H}}_-$ and $\hat{H}_+ = \hat{T}\hat{\mathcal{H}}_+\hat{T}^\dagger$, where $\hat{\mathcal{H}}_\pm = \hat{B}_\mp \hat{B}_\pm$ and the shape invariance condition can be written as the commutation relation $[\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger R(a_1)\hat{T} \equiv R(a_0)$, suggesting that $\hat{B}_-$ and $\hat{B}_+$ are the appropriate creation and annihilation operators for the spectra of the shape-invariant potential systems provided that their non-commutativity with $R(a_n)$ is taken into account. Indeed, using relations $\hat{B}_\pm R(a_n) = R(a_{n\pm 1})\hat{B}_\pm$ which readily follow from the definitions of $\hat{B}_\pm$, one gets the commutation relations

$$\underbrace{[\hat{B}_+, [\hat{B}_+, [\hat{B}_+, \ldots, [\hat{B}_+, [\hat{B}_+, R(a_0)]] \ldots]]]}_{\text{sequence of } n \text{ commutation operations}} = \left\{\sum_{k=0}^{n} (-1)^k \binom{n}{k} R(a_{n-k})\right\} \hat{B}_+^n \tag{7}$$

where we used the binomial coefficient definition: $\binom{n}{k} = n!/\{k!(n-k)!\}$. There are an infinite number of these commutation relations that, with their adjoint commutation relations and the commutator $[\hat{B}_-, \hat{B}_+] = R(a_0)$ form an infinite-dimensional Lie algebra, realized here in a unitary representation.

The ground state of the Hamiltonian $\hat{\mathcal{H}}_-$ satisfies the condition $\hat{A}|0\rangle = 0 = \hat{B}_-|0\rangle$. Using this fact and the relations above it is possible to obtain the eigenvalue equations $\hat{\mathcal{H}}_-|n\rangle = e_n|n\rangle$ and $\hat{\mathcal{H}}_+|n\rangle = \{e_n + R(a_0)\}|n\rangle$ where the normalized nth excited eigenstate $|n\rangle = \hat{K}_+^n|0\rangle$ can be obtained from the ground state by the successive action of the raising operator $\hat{K}_+ \equiv \frac{1}{\sqrt{\hat{\mathcal{H}}_-}}\hat{B}_+$ and the related eigenvalues are given by $e_0 = 0$ and $e_n = \sum_{k=1}^{n} R(a_k)$, for $n \geqslant 1$. With the results above it is possible to show that

$$\hat{B}_+|n\rangle = \sqrt{e_{n+1}}|n+1\rangle \qquad \text{and} \qquad \hat{B}_-|n\rangle = \sqrt{e_{n-1} + R(a_0)}|n-1\rangle, \tag{8}$$

making clear the ladder nature of the operators $\hat{B}_\pm$ when applied on the eigenstates $\{|n\rangle; n = 0, 1, 2, \ldots\}$ of $\hat{\mathcal{H}}_-$.

Shape invariance of self-similar potentials was studied in detail in [29, 30]. In the simplest case studied the remainder in the shape invariance condition is given by $R(a_1) = ca_1$, where c is a constant. Using this expression and the scaling relation $a_n = q^n a_0$ we can show that the eigenvalues of the system have the form

$$e_n = \sum_{k=1}^{n} R(a_k) = \left(\frac{1-q^n}{1-q}\right) R(a_1). \tag{9}$$

2.2. Purely coherent and squeezed states

The purely coherent states are considered to be the quantum states which most closely approach the classical limit. They can be defined by three different methods that are equivalent only in the special case of the harmonic oscillator. According to the ladder-operator method, a purely coherent state $|z; a_j\rangle_{\mathrm{C}}$ for shape-invariant systems can be obtained by the Glauber expansion [26, 31]

$$|z; a_j\rangle_{\mathrm{C}} = \mathcal{N}_{\mathrm{C}}(\rho_{\mathrm{o}}, a_j) \sum_{n=0}^{\infty} \frac{z^n}{h_n^{(\mathrm{C})}(a_j)}|n\rangle,$$
$$\frac{1}{\mathcal{N}_{\mathrm{C}}(\rho_{\mathrm{o}}, a_j)} = \sqrt{\sum_{n=0}^{\infty} \frac{\rho_{\mathrm{o}}^n}{|h_n^{(\mathrm{C})}(a_j)|^2}}, \qquad \text{with} \qquad \rho_{\mathrm{o}} = |z|^2 \tag{10}$$

and satisfies the eigenvalue equation $\hat{B}_-|z; a_j\rangle_{\mathrm{C}} = z\mathcal{Z}_{j-1}\{\mathcal{N}_{\mathrm{C}}(\rho_{\mathrm{o}}, a_{j-1})/\mathcal{N}_{\mathrm{C}}(\rho_{\mathrm{o}}, a_j)\}|z; a_j\rangle_{\mathrm{C}}$ where $\mathcal{N}_{\mathrm{C}}(\rho_{\mathrm{o}}, a_{j-1}) = \hat{T}^\dagger \mathcal{N}_{\mathrm{C}}(\rho_{\mathrm{o}}, a_j)\hat{T}$, with $h_0^{(\mathrm{C})}(a_j) = 1$ and

$$h_n^{(\mathrm{C})}(a_j) = \prod_{k=0}^{n-1}\{\sqrt{e_n - e_k}/\mathcal{Z}_{j+k}\} = \prod_{k=1}^{n}\left\{\frac{1}{\mathcal{Z}_{j+k-1}}\sqrt{\sum_{s=k}^{n} R(a_s)}\right\}, \qquad \text{for} \qquad n \geqslant 1. \tag{11}$$

In this expression $\mathcal{Z}_{j+k} = \hat{T}^k \mathcal{Z}_j \hat{T}^{\dagger k}$, where $\mathcal{Z}_j \equiv \mathcal{Z}(a_1, a_2, a_3, \ldots)$ is a general complex functional.

Again in the ladder-operator method, a purely squeezed state $|z; a_j\rangle_{\mathrm{S}}$ for shape-invariant systems can be obtained, in a generalized way, by the Glauber expansion [26]

$$|z; a_j\rangle_{\mathrm{S}} = \mathcal{N}_{\mathrm{S}}(\rho_{\mathrm{o}}, a_j) \sum_{n=0}^{\infty} \frac{z^n}{h_n^{(\mathrm{S})}(a_j)}|2n\rangle, \qquad \frac{1}{\mathcal{N}_{\mathrm{S}}(\rho_{\mathrm{o}}, a_j)} = \sqrt{\sum_{n=0}^{\infty} \frac{\rho_{\mathrm{o}}^n}{|h_n^{(\mathrm{S})}(a_j)|^2}} \tag{12}$$

and satisfies the definition eigenvalue equation $\hat{B}_-|z;a_j\rangle_{\rm S} = z\mathcal{Z}_{j-1}\{\mathcal{N}_{\rm S}(\rho_{\rm o},a_{j-1})/\mathcal{N}_{\rm S}(\rho_{\rm o},a_{j+1})\}\hat{B}_+|z;a_j\rangle_{\rm S}$ where $\mathcal{N}_{\rm S}(\rho_{\rm o},a_{j+1}) = \hat{T}\mathcal{N}_{\rm S}(\rho_{\rm o},a_j)\hat{T}^\dagger$, with $h_0^{(\rm S)}(a_j) = 1$ and

$$h_n^{(\rm S)}(a_j) = \prod_{k=0}^{n-1}\left\{\sqrt{\frac{e_{2n}-e_{2k}}{e_{2n}-e_{2k+1}}}\Big/\mathcal{Z}_{j+2k}\right\} = \prod_{k=1}^{n}\left\{\frac{1}{\mathcal{Z}_{j+2k-2}}\sqrt{\left[\sum_{s=2k-1}^{2n}R(a_s)\Big/\sum_{s=2k}^{2n}R(a_s)\right]}\right\},$$
$$\text{for} \qquad n\geqslant 1. \tag{13}$$

3. Investigating squeezing effects

The uncertainty principle limits the precise knowledge of all physical quantities in a quantum system and one tool often used to overcome such restrictions in practical applications is the squeezed states. Today, because of their intrinsic properties, the squeezed states have found potential applications in high-precision optical measurements, optical communications, detection of weak signals, atomic and molecular physics, and other quantum physics. We investigate in this section the squeezing effects observed in a self-similar potential system via two different approaches: one when the quantum state of the system is associated with a purely squeezed state and another when it happens with a superposition of purely coherent states.

3.1. Squeezing via purely squeezed states

From the ladder operators $\hat{B}_\pm$ we can introduce two generalized quadrature operators $\hat{X} = (\hat{B}_+ + \hat{B}_-)/\sqrt{2}$ and $\hat{P} = {\rm i}(\hat{B}_+ - \hat{B}_-)/\sqrt{2}$ which satisfy the canonical commutation relation $[\hat{X},\hat{P}] = {\rm i}R(a_0)$. The squeezing property of the state $|z;a_j\rangle_{\rm S}$ can be evaluated by calculating the variances of the quadrature operators $\sigma_X^{(\rm S)}$ and $\sigma_P^{(\rm S)}$ in this state. We use $\sigma_O^{(\rm S)} \equiv (\Delta\hat{O})^2_{\rm S} \equiv {}_{\rm S}\langle z;a_j|(\hat{O}-\langle\hat{O}\rangle_{\rm S})^2|z;a_j\rangle_{\rm S} = \langle\hat{O}^2\rangle_{\rm S} - \langle\hat{O}\rangle^2_{\rm S}$, where the notation $\langle\hat{O}\rangle_{\rm S} = {}_{\rm S}\langle z;a_j|\hat{O}|z;a_j\rangle_{\rm S}$ stands for the expectation value of a given observable $\hat{O}$ in the state $|z;a_j\rangle_{\rm S}$ of the quantum system. Using the relations (8) and definition (12) we find that $\langle\hat{X}\rangle_{\rm S} = {}_{\rm S}\langle z;a_j|\hat{X}|z;a_j\rangle_{\rm S} = 0 = {}_{\rm S}\langle z;a_j|\hat{P}|z;a_j\rangle_{\rm S} = \langle\hat{P}\rangle_{\rm S}$ and that

$$\sigma_X^{(\rm S)}(z,a_j) = {}_{\rm S}\langle z;a_j|\hat{X}^2|z;a_j\rangle_{\rm S} = \{1+2\eta_{\rm S}(z,a_j) + [1+\eta_{\rm S}(z,a_j)]\delta_{\rm S}(z,a_j)\}\Delta_{\rm H} \tag{14}$$

$$\sigma_P^{(\rm S)}(z,a_j) = {}_{\rm S}\langle z;a_j|\hat{P}^2|z;a_j\rangle_{\rm S} = \{1+2\eta_{\rm S}(z,a_j) - [1+\eta_{\rm S}(z,a_j)]\delta_{\rm S}(z,a_j)\}\Delta_{\rm H}, \tag{15}$$

where $\Delta_{\rm H} \equiv \frac{1}{2}|\langle[\hat{X},\hat{P}]\rangle_{\rm S}| = \frac{1}{2}R(a_0)$ gives the Heisenberg uncertainty minimum value as a reference and the other related factors are obtained by

$$\begin{aligned} &\delta_{\rm S}(z,a_j) = 2\left(\frac{\mathcal{N}_{\rm S}(\rho_{\rm o},a_{j-2})}{\mathcal{N}_{\rm S}(\rho_{\rm o},a_j)}\right){\rm Re}\{z\mathcal{Z}_{j-2}\} \qquad \text{and} \\ &\eta_{\rm S}(z,a_j) = \mathcal{N}^2_{\rm S}(\rho_{\rm o},a_j)\sum_{n=0}^{\infty}\frac{e_{2n}}{R(a_0)}\left(\frac{\rho_{\rm o}^n}{|h_n^{(\rm S)}(a_j)|^2}\right)\end{aligned} \tag{16}$$

with $\mathcal{N}_{\rm S}(\rho_{\rm o},a_{j-2}) = \hat{T}^{\dagger 2}\mathcal{N}_{\rm S}(\rho_{\rm o},a_j)\hat{T}^2$. Using expression (9) of e_n for self-similar potentials, we find that

$$\prod_{s=0}^{n-1}\sqrt{\frac{e_{2n}-e_{2s}}{e_{2n}-e_{2s+1}}} = \sqrt{\frac{(q^2;q^2)_n}{q^n(q;q^2)_n}}, \tag{17}$$

where the q-shifted factorial $(q;q)_n$ is defined as $(p;q)_0 = 1$ and $(p;q)_n = \prod_{j=0}^{n-1}(1-pq^j)$ with $n\in\mathbb{Z}$ and $n\geqslant 1$. Therefore, if we assume the generalizing functional with the form

$\mathcal{Z}_j = \sqrt{R(a_2)/(1-ba_3)}$, where b is a real constant, and use this propose and the scaling relation $a_n = a_0 q^n$ we have that $\mathcal{Z}_{j+2k-2} \equiv \hat{T}^{(2k-2)} \mathcal{Z}_j \hat{T}^{\dagger(2k-2)} = \sqrt{R(a_1) q^{2k-1}/(1-ba_1 q^{2k})}$ and thus we can prove that

$$\prod_{k=1}^{n} \mathcal{Z}_{j+2k-2} = \sqrt{\frac{(1-ba_1)[R(a_1)]^n q^{n^2}}{(ba_1;q^2)_{n+1}}}. \tag{18}$$

Assuming that $ba_1 = 1/q$ and inserting relations (17) and (18) into equation (13) we get

$$h_n^{(\mathrm{S})}(a_j) = \sqrt{\frac{(q^2;q^2)_n}{[R(a_1)]^n q^{n(n+1)}}} \tag{19}$$

where we used the q-shifted factorial relation $(a;q)_{n+k} = (a;q)_k (aq^k;q)_n$. Inserting expression (19) into (12) we find

$$\frac{1}{\mathcal{N}_{\mathrm{S}}^2(\rho_{\mathrm{o}}, a_j)} = \sum_{n=0}^{\infty} \frac{[\rho_{\mathrm{o}} R(a_1)]^n q^{n(n+1)}}{(q^2;q^2)_n} = E_{q^2}(q^2\rho) = (-q^2\rho;q^2)_\infty \qquad \text{where} \qquad \rho \equiv \rho_{\mathrm{o}} R(a_1) \tag{20}$$

and we used the Jackson's q-exponential definition [32] $E_q(z) = \sum_{n=0}^{\infty} q^{n(n-1)/2} z^n/(q;q)_n = (-z;q)_\infty$. Following a similar procedure and using the expression of the generalizing functional $\mathcal{Z}_j$ and result (17) it is possible to show after some calculations, that

$$h_n^{(\mathrm{S})}(a_{j-2}) = \hat{T}^{\dagger 2} h_n^{(\mathrm{S})}(a_j) \hat{T}^2 = \sqrt{\frac{(q^2;q^2)_n (1-1/q)}{[R(a_1)]^n q^{n(n-1)} (1-q^{2n-1})}} \tag{21}$$

and thus

$$\frac{1}{\mathcal{N}_{\mathrm{S}}^2(\rho_{\mathrm{o}}, a_{j-2})} = \frac{1}{1-q}[q(-\rho;q^2)_\infty - (-q^2\rho;q^2)_\infty]. \tag{22}$$

We can then show that the factors in (16) related to with the variances of the quadrature operators are given by

$$\Delta_{\mathrm{H}} = \frac{R(a_1)}{2q}, \qquad \delta_{\mathrm{S}}(\rho, q) = 2\sqrt{\frac{R(a_1)}{1 - q(-\rho;q^2)_\infty/(-q^2\rho;q^2)_\infty}} \operatorname{Re} z, \qquad \eta_{\mathrm{S}}(\rho, q) = \frac{q}{1-q}\left[1 - \frac{(-q^4\rho;q^2)_\infty}{(-q^2\rho;q^2)_\infty}\right] \tag{23}$$

where we used that $e_{2n}/R(a_0) = q(1-q^{2n})/(1-q)$.

3.2. Squeezing via the superposition of purely coherent states

The linear superposition principle is one of the most fundamental features of quantum mechanics. It was realized that the interference between quantum states gives rise to various non-classical effects [33]. In particular, it was shown that the squeezing phenomenon emerges from a superposition of coherent states [34]. Following this idea, we investigate squeezing in self-similar potential systems by the superposition of two purely coherent states along a straight line on the z-plane [35]. The states constructed in this case, called *even coherent states* $|z;a_j\rangle_+$ and *odd coherent states* $|z;a_j\rangle_-$, are given by

$$|z;a_j\rangle_\pm = C_\pm(\rho_{\mathrm{o}};a_j)\{|z;a_j\rangle_{\mathrm{C}} \pm |-z;a_j\rangle_{\mathrm{C}}\} \tag{24}$$

where, using expression (10), we can prove that the normalization condition ${}_{\pm}\langle z; a_j|z; a_j\rangle_{\pm} = 1$ is satisfied when

$$C_{\pm}(\rho_{\rm o}; a_j) = \sqrt{\frac{\mathcal{N}_{\rm C}^2(-\rho_{\rm o}, a_j)}{2\left\{\mathcal{N}_{\rm C}^2(-\rho_{\rm o}, a_j) \pm \mathcal{N}_{\rm C}^2(\rho_{\rm o}, a_j)\right\}}}. \tag{25}$$

We observe that for large values of the amplitudes, $|z| \gg 1$, the components $|\pm z; a_j\rangle_{\rm C}$ become macroscopically distinguishable and we get with such a superposition (24) two examples of the so-called *Schrödinger cat state* [36]. One can show that ${}_{\pm}\langle z; a_j|z; a_j\rangle_{\mp} = 0$, and we find that

$$\hat{B}_{-}|z; a_j\rangle_{\pm} = z\mathcal{Z}_{j-1}\left[\frac{\mathcal{N}_{\rm C}(\rho_{\rm o}, a_{j-1})C_{\pm}(\rho_{\rm o}; a_{j-1})}{\mathcal{N}_{\rm C}(\rho_{\rm o}, a_j)C_{\mp}(\rho_{\rm o}; a_j)}\right]|z; a_j\rangle_{\mp} \tag{26}$$

$$\hat{B}_{-}^2|z; a_j\rangle_{\pm} = \rho\mathcal{Z}_{j-1}\mathcal{Z}_{j-2}\left[\frac{\mathcal{N}_{\rm C}(\rho_{\rm o}, a_{j-2})C_{\pm}(\rho_{\rm o}; a_{j-2})}{\mathcal{N}_{\rm C}(\rho_{\rm o}, a_j)C_{\pm}(\rho_{\rm o}; a_j)}\right]|z; a_j\rangle_{\pm}. \tag{27}$$

One then gets $\langle\hat{X}\rangle_{\pm} = {}_{\pm}\langle z; a_j|\hat{X}|z; a_j\rangle_{\pm} = 0 = {}_{\pm}\langle z; a_j|\hat{P}|z; a_j\rangle_{\pm} = \langle\hat{P}\rangle_{\pm}$. It is possible to observe squeezing effects in the even coherent state $|z; a_j\rangle_{+}$ and we find that

$$\sigma_X^{(+)}(z, a_j) = {}_{+}\langle z; a_j|\hat{X}^2|z; a_j\rangle_{+} = \{1 + 2\eta_{+}(z, a_j) + 2\delta_{+}(z, a_j)\}\Delta_{\rm H} \tag{28}$$

$$\sigma_P^{(+)}(z, a_j) = {}_{+}\langle z; a_j|\hat{P}^2|z; a_j\rangle_{+} = \{1 + 2\eta_{+}(z, a_j) - 2\delta_{+}(z, a_j)\}\Delta_{\rm H}, \tag{29}$$

where

$$\eta_{+}(z, a_j) = \frac{\rho_{\rm o}\mathcal{Z}_{j-1}^2}{R(a_0)}\left[\frac{\mathcal{N}_{\rm C}^2(\rho_{\rm o}, a_{j-1})C_{+}^2(\rho_{\rm o}; a_{j-1})}{\mathcal{N}_{\rm C}^2(\rho_{\rm o}, a_j)C_{-}^2(\rho_{\rm o}; a_j)}\right] \tag{30}$$

$$\delta_{+}(z, a_j) = \frac{\rho_{\rm o}\mathcal{Z}_{j-1}\mathcal{Z}_{j-2}}{R(a_0)}\left[\frac{\mathcal{N}_{\rm C}(\rho_{\rm o}, a_{j-2})C_{+}(\rho_{\rm o}; a_{j-2})}{\mathcal{N}_{\rm C}(\rho_{\rm o}, a_j)C_{+}(\rho_{\rm o}; a_j)}\right] \tag{31}$$

when we have a real functional $\mathcal{Z}_j$. Using expression (9) for e_n for the self-similar potentials, we find that

$$\prod_{k=0}^{n-1}\sqrt{e_n - e_k} = \sqrt{\frac{[R(a_1)]^n q^{n(n-1)/2}(q; q)_n}{(1-q)^n}} \tag{32}$$

and, if we assume $\mathcal{Z}_j = R(a_1)$ and use the scaling relation $a_n = a_0 q^n$, we get $\mathcal{Z}_{j+k-1} \equiv \hat{T}^{(k-1)}\mathcal{Z}_j\hat{T}^{\dagger(k-1)} = R(a_1)q^{k-1}$ and thus we can prove that

$$\prod_{k=1}^{n}\mathcal{Z}_{j+k-1} = [R(a_1)]^n q^{n(n-1)/2}. \tag{33}$$

Therefore, using relations (32) and (33) into equation (11) we find that

$$h_n^{(\rm C)}(a_j) = \sqrt{\frac{(q; q)_n}{[R(a_1)]^n(1-q)^n q^{n(n-1)/2}}} \tag{34}$$

and thus, using (34) and (11), it follows that

$$\frac{1}{\mathcal{N}_{\rm C}^2(\rho_{\rm o}, a_j)} = \sum_{n=0}^{\infty}\frac{q^{n^2/2}}{(q; q)_n}[\rho_{\rm o}R(a_1)(1-q)/\sqrt{q}]^n = E_q^{(1/2)}\left(\xi_q^{(0)}\right) \tag{35}$$
$$\text{where} \qquad \xi_q^{(0)} \equiv \rho(1-q)/\sqrt{q}$$

and we used the q-exponential definition [37] $E_q^{(\mu)}(z) = \sum_{n=0}^{\infty} q^{\mu n^2} z^n/(q;q)_n$. Following a similar procedure and using the expression of the functional $\mathcal{Z}_j$ and result in (32) it is possible to show after some calculations that

$$h_n^{(\mathrm{C})}(a_{j-k}) = \hat{T}^{\dagger k} h_n^{(\mathrm{C})}(a_j)\hat{T}^k = \sqrt{\frac{(q;q)_n}{[R(a_1)]^n(1-q)^n q^{n(n-2k-1)/2}}}, \tag{36}$$

and thus

$$\frac{1}{\mathcal{N}_{\mathrm{C}}^2(\rho_{\mathrm{o}}, a_{j-k})} = E_q^{(1/2)}\left(\xi_q^{(k)}\right) \qquad \text{where} \qquad \xi_q^{(k)} = \xi_q^{(0)}/q^k. \tag{37}$$

In this case, the factors in equation (16) related to the variances of the quadrature operators are given by

$$\eta_+(\rho, q) = \frac{\rho}{q}\left(\frac{r_q^{(0)}-1}{r_q^{(1)}+1}\right) \qquad \text{and} \qquad \delta_+(\rho, q) = \frac{\rho}{q^2}\sqrt{\frac{E_q^{(1/2)}\left(-\xi_q^{(0)}\right)}{E_q^{(1/2)}\left(-\xi_q^{(2)}\right)}\left(\frac{r_q^{(0)}+1}{r_q^{(2)}+1}\right)} \tag{38}$$

where $r_q^{(k)} \equiv E_q^{(1/2)}\left(\xi_q^{(k)}\right)/E_q^{(1/2)}\left(-\xi_q^{(k)}\right)$.

In order to compare the squeezing effects in self-similar potential systems obtained with the purely squeezed state $|z; a_j\rangle_{\mathrm{S}}$ and the purely coherent composite state $|z; a_j\rangle_+$ constructed in this section, we display in figure 1 the three-dimensional plots of the surfaces of the variances $\sigma_P^{(S)}(\rho, q)$ and $\sigma_P^{(+)}(\rho, q)$, measured in units of $\Delta_{\mathrm{H}} = 0.5R(a_1)/q$, as a function of the quantum state expansion factor ρ and the self-similar scaling parameter q, presented in a logarithmic scale. In all numerical calculations presented in the figures of this paper we assumed the remainder value to be $R(a_1) = 1$. It should be noted from the figure that we obtain squeezing effects in the variance of $\hat{P}$, characterized by elevated regions of the σ_P-surface with respect to the unphysical bottom plane. However, the behaviour of the variance of $\hat{P}$ in terms of ρ and q is different for each case. The variance $\sigma_P^{(S)}(\rho, q)$ is restricted in general to the region with $\log\rho < -0.5$ and shows a weak dependence with q when $\log q < -1.0$. For the range $-0.5 < \log q \leqslant 1$ the squeezing surface of $\sigma_P^{(S)}(\rho, q)$ shows decreasing behaviour restricting the allowed values for ρ further. On the other hand, the variance $\sigma_P^{(+)}(\rho, q)$, besides the common property of being present in a restricted region in the $(q\rho)$-plane, shows very different behaviour in terms of these variables. The squeezing surface of $\sigma_P^{(+)}(\rho, q)$ has an almost triangular plateau formed around the $(\log\rho = -4, q = 1)$-corner, with an increasing border with respect to the unphysical bottom plane given by the approximated expression $\log\rho = \log q^2 - 0.2$. To conclude, it should be emphasized that the self-similar potential system shows qualitative deviations from the Heisenberg's minimal uncertainty relation, which in this case [26] is given by $\sigma_X\sigma_P \geqslant \frac{1}{2}R(a_1)$. Like some special states for quantum deformed systems [38–40], we observe that with the purely squeezed state $|z; a_j\rangle_{\mathrm{S}}$ or the purely coherent composite state $|z; a_j\rangle_+$, it is possible to have the variance in a given quadrature observable approaching zero (around the base of the plateau region) while the variance in the other observable remains finite. Indeed, this is an expected result taking into account the relationship between the self-similar potential system and q-deformed models for the harmonic oscillator system [18, 30]. The deviations from the Heisenberg minimal uncertainty relation are accepted for distances much smaller than the limit of the quantum electrodynamics theories according to the experimental tests [39]. The q-deformed quantum mechanics is a possible candidate to explain theoretically this short distance physics [39, 41].

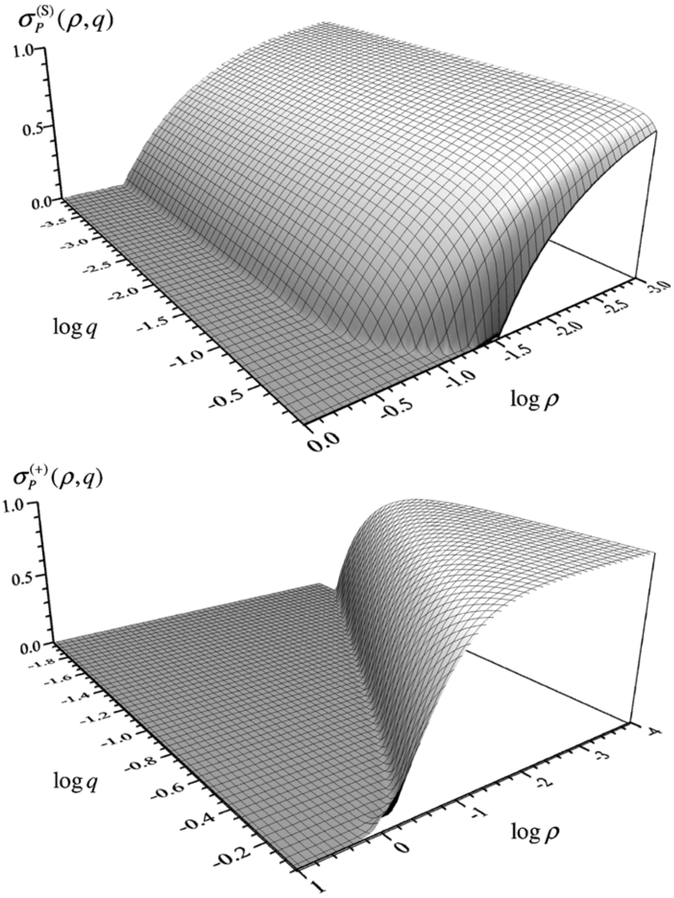

Figure 1. Three-dimensional graphics of the variances $\sigma_P^{(S)}(\rho, q)$ and $\sigma_P^{(+)}(\rho, q)$, measured in units of $\Delta_{\mathrm{H}} = 0.5R(a_1)/q$, in terms of the quantum state expansion factor ρ and the self-similar scaling parameter q, presented in logarithmic scales.

4. Investigating thermal effects

Setting $\epsilon = 1 - q \ll 1$ in equation (9) for a given eigenvalue e_n of the self-similar potential we can write the expansion

$$e_n(\epsilon) = \left[\frac{1-(1-\epsilon)^n}{\epsilon}\right] R(a_1) \approx \left\{\left[1 + \frac{\epsilon}{2} + \frac{\epsilon^2}{3} + \frac{\epsilon^3}{4} + \cdots\right] n - \frac{\epsilon}{2!}\left[1 + \epsilon + \frac{11}{12}\epsilon^2 + \cdots\right] n^2 + \frac{\epsilon^2}{3!}\left[1 + \frac{3}{2}\epsilon + \cdots\right] n^3 + \mathcal{O}(n^4)\right\} R(a_1) \tag{39}$$

showing that the $\epsilon \to 0$ limit describes the harmonic oscillator potential case $e_n = nR(a_1)$ and, in general, the eigenvalue e_n for a self-similar potential incorporates terms in powers of n that can be interpreted as anharmonic contributions in the ordinary harmonic oscillator potential [24]. Another possible interpretation of the expansion terms of e_n is associated with the q-boson interacting models where now the $\epsilon \to 0$ limit describes the non-interacting particles (corresponding to an undeformed free gas) while the higher order terms in powers of ϵ represent interaction contributions (corresponding to a q-deformed free gas) [25]. In either case it would be interesting to study quantum statistics of an ensemble constituted with this kind of systems in terms of the scaling parameter q. In order to carry out this study, we consider a quantum gas of self-similar potential systems in thermodynamic equilibrium with the reservoir at temperature Θ, which obeys the quantum canonical distribution. The corresponding density operator in this case reads as

$$\hat{\varrho}_\beta = \frac{1}{Z_\beta}\sum_{n=0}^{\infty} \mathrm{e}^{-\beta e_n}|n\rangle\langle n|, \tag{40}$$

where $\beta = 1/\Theta$ is the Boltzmann's temperature coefficient (we assume the Boltzmann's constant value $k_{\mathrm{B}} = 1$) and the partition function of the system is obtained by

$$Z_\beta \equiv Z(\beta, q) = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta e_n} = \mathrm{e}^{-\alpha_q}\sum_{n=0}^{\infty} \mathrm{e}^{\alpha_q q^n} \qquad \text{where} \qquad \alpha_q = \frac{\beta R(a_1)}{1-q}. \tag{41}$$

4.1. Thermal effects via purely coherent states

From both the physical and mathematical points of view, purely coherent and purely squeezed states of quantum mechanics are fascinating objects having important applications in many fields. Often one is interested in the evaluation of the moments $\{\langle \hat{O}^k\rangle_{\mathrm{C,S}}, k = 1, 2, 3, \ldots\}$ of some observable $\hat{O}$ when the state of the system is represented by these special quantum states. From a practical point of view, the most important observables are related to the powers of $\hat{\mathcal{H}}_-$. In general, overcompleteness of the coherent states makes it possible to obtain useful information about a given observable from its diagonal expansion in these quantum states. In this sense, taking into account equation (10) and the relation $\langle n|\hat{\mathcal{H}}_-^k|n'\rangle = e_n^k\delta_{nn'}$ we find that

$$\begin{aligned}\langle \hat{\mathcal{H}}_-^k(\rho, q)\rangle_{\mathrm{C}} \equiv {}_{\mathrm{C}}\langle z; a_j|\hat{\mathcal{H}}_-^k|z; a_j\rangle_{\mathrm{C}} &= \mathcal{N}_{\mathrm{C}}^2(\rho_{\mathrm{o}}, a_j)\sum_{n=0}^{\infty}\frac{e_n^k\rho_{\mathrm{o}}^n}{\left|h_n^{(\mathrm{C})}(a_j)\right|^2} \\ &= \frac{1}{E_q^{(1/2)}\big(\xi_q^{(0)}\big)}\sum_{n=0}^{\infty}\frac{e_n^k q^{n^2/2}\big[\xi_q^{(0)}\big]^n}{(q;q)_n}\end{aligned} \tag{42}$$

where we used equations (34) and (35). Considering relation (9) and the definition of the q-exponential function $E_q^{(\mu)}(z)$, we can prove after some calculations that

$$\sum_{n=0}^{\infty}\frac{e_n^k q^{n^2/2}\big[\xi_q^{(0)}\big]^n}{(q;q)_n} = \left[\frac{R(a_1)}{1-q}\right]^k\sum_{m=0}^{k}(-1)^m\binom{k}{m}E_q^{(1/2)}\big(\xi_q^{(-m)}\big) \tag{43}$$

and with this result in (42) it follows that

$$\langle \hat{\mathcal{H}}_-^k(\rho, q)\rangle_{\mathrm{C}} = \left[\frac{R(a_1)}{1-q}\right]^k\sum_{m=0}^{k}(-1)^m\binom{k}{m}\chi_q^{(-m)} \qquad \text{where} \qquad \chi_q^{(j)} \equiv \frac{E_q^{(1/2)}\big(\xi_q^{(j)}\big)}{E_q^{(1/2)}\big(\xi_q^{(0)}\big)}, \tag{44}$$

showing that the expectation value of $\hat{\mathcal{H}}_-^k$ in the purely coherent state $|z; a_j\rangle_{\mathrm{C}}$ can be obtained through the sum of $(k+1)$ parcels in the ratio factor $\chi_q^{(-m)}$ with binomial coefficients $\binom{k}{m}$. These expectation values are useful in order to calculate the correlation functions that provide information about the inherent statistical properties of the state $|z; a_j\rangle_{\mathrm{C}}$. One of these quantities is the Mandel parameter [42], defined as $Q^{(\mathrm{M})} \equiv (\Delta\hat{\mathcal{H}}_-)^2/\langle\hat{\mathcal{H}}_-\rangle - 1$ for a shape-invariant potential system in general. In the case of a self-similar potential system in a purely coherent state, using the equivalent expression $Q_{\rm C}^{(\rm M)} \equiv \langle \hat{h}_{-}\rangle_{\rm C}\{[\langle \hat{h}_{-}^2\rangle_{\rm C} - \langle \hat{h}_{-}\rangle_{\rm C}]/\langle \hat{h}_{-}\rangle_{\rm C}^2 - 1\}$, with $\hat{h}_{-} \equiv \hat{\mathcal{H}}_{-}/R(a_1)$, we find that

$$Q_{\rm C}^{(\rm M)}(\rho, q) = \frac{1}{1-q}\left\{q\left(\frac{1-\chi_q^{(-2)}}{1-\chi_q^{(-1)}}\right) + \chi_q^{(-1)} - 1\right\}. \tag{45}$$

The Mandel parameter $Q_{\rm C}^{(\rm M)}(\rho, q)$ offers the information on the character of the purely coherent state $|z; a_j\rangle_{\rm C}$ versus the Poisson distribution function, characteristic of the ordinary harmonic oscillator system. Since the self-similar potential can be assumed with the addition of anharmonic contribution in the ordinary harmonic oscillator potential system, it is interesting to recall that for the harmonic oscillator system, the purely coherent state exhibits a Poissonian distribution and we have $Q^{(\rm M)} = 0$. In our case, if we find $Q_{\rm C}^{(\rm M)} < 0$ (> 0) we can say that the state $|z; a_j\rangle_{\rm C}$ exhibits a sub-Poissonian (super-Poissonian) statistic.

In order to examine other interesting properties of the purely coherent state $|z; a_j\rangle_{\rm C}$ for the self-similar potential associated with mixed quantum states, i.e. the thermal states, we consider the diagonal matrix elements of the density operator $\hat{\varrho}_\beta$ in the purely coherent state, $Q_{\rm C}^{(\rm H)}(\beta; z, a_j) \equiv {}_{\rm C}\langle z; a_j|\hat{\varrho}_\beta|z; a_j\rangle_{\rm C}$, which defines the Husimi's Q-function in this representation. Phase-space techniques are very useful in studying quantum mechanics and quantum statistics [43, 44]. Among the various phase distributions the Husimi function is the most popular and is used in the study of quantum fluctuation [45], quantum entanglement and quantum information [46], and as a measure of the complexity of quantum states [47]. In our case, when we use results (34) and (35) we find that

$$Q_{\rm C}^{(\rm H)}(\beta; \rho_{\rm o}, q) = \frac{\mathcal{N}_{\rm C}^2(\rho_{\rm o}, a_j)}{Z_\beta}\sum_{n=0}^{\infty}\frac{{\rm e}^{-\beta e_n}\rho_{\rm o}^n}{\left|h_n^{(\rm C)}(a_j)\right|^2} = \frac{1}{Z_\beta E_q^{(1/2)}\left(\xi_q^{(0)}\right)}\sum_{n=0}^{\infty}\frac{{\rm e}^{-\beta e_n}q^{n^2/2}\left[\xi_q^{(0)}\right]^n}{(q;q)_n}. \tag{46}$$

Figure 2 displays, in a set of six stages, the thermal evolution of the Husimi $Q_{\rm C}^{(\rm H)}$-surface as a function of the quantum state expansion factor ρ and the temperature coefficient β. To help the visualization of the $Q_{\rm C}^{(\rm H)}$-surface behaviour, we show in each small figure constant level curves. Following the sequence of small figures with the increasing temperature coefficient β, we observe that: (i) $Q_{\rm C}^{(\rm H)}(\beta; \rho, q)$ starts from ρ-low sensitivity behaviour for very low values of β and changes for q-low sensitivity behaviour for very high values of β; (ii) for intermediate values of β and $\rho > 1$ the $Q_{\rm C}^{(\rm H)}$-surface folds in a maximum-value line that moves closer to the ρ-axis when β increases; (iii) as expected, for any value of β the positive bounded condition $0 \leqslant Q_{\rm C}^{(\rm H)} < 1$ is always satisfied .

It is not difficult to verify that the normalization condition of the density operator

$$\operatorname{Tr}\hat{\varrho}_\beta = \int {\rm d}^2z\, w_{\rm C}(\rho_{\rm o}, q)Q_{\rm C}^{(\rm H)}(\beta; \rho_{\rm o}, q) = 1 \tag{47}$$

is satisfied if we assume the existence of the positive-definite weight function

$$w_{\rm C}(\rho_{\rm o}, q) = \frac{R(a_1)(1-q)}{\pi q\log(1/q)}\frac{\left(-q^{\frac{1}{2}}\xi_q^{(0)}; q\right)_\infty}{\left(-q^{\frac{1}{2}}\xi_q^{(1)}; q\right)_\infty} \tag{48}$$

and use the relation $E_q^{(1/2)}\left(\xi_q^{(0)}\right) = \left(-q^{\frac{1}{2}}\xi_q^{(0)}; q\right)_\infty$ together with the Ramanujan's integral [48]

$$\int_0^\infty {\rm d}t\, t^n\frac{(-at; q)_\infty}{(-t; q)_\infty} = \frac{\log(1/q)(q; q)_n}{q^{n(n+1)/2}(a; q^{-1})_{n+1}}. \tag{49}$$

(An elementary proof of the Ramanujan's integral is given by Askey [49].)

Since the purely coherent states $|z; a_j\rangle_{\rm C}$ for a self-similar potential system form an overcomplete set of states, they can be used as a basis set despite the fact that they are

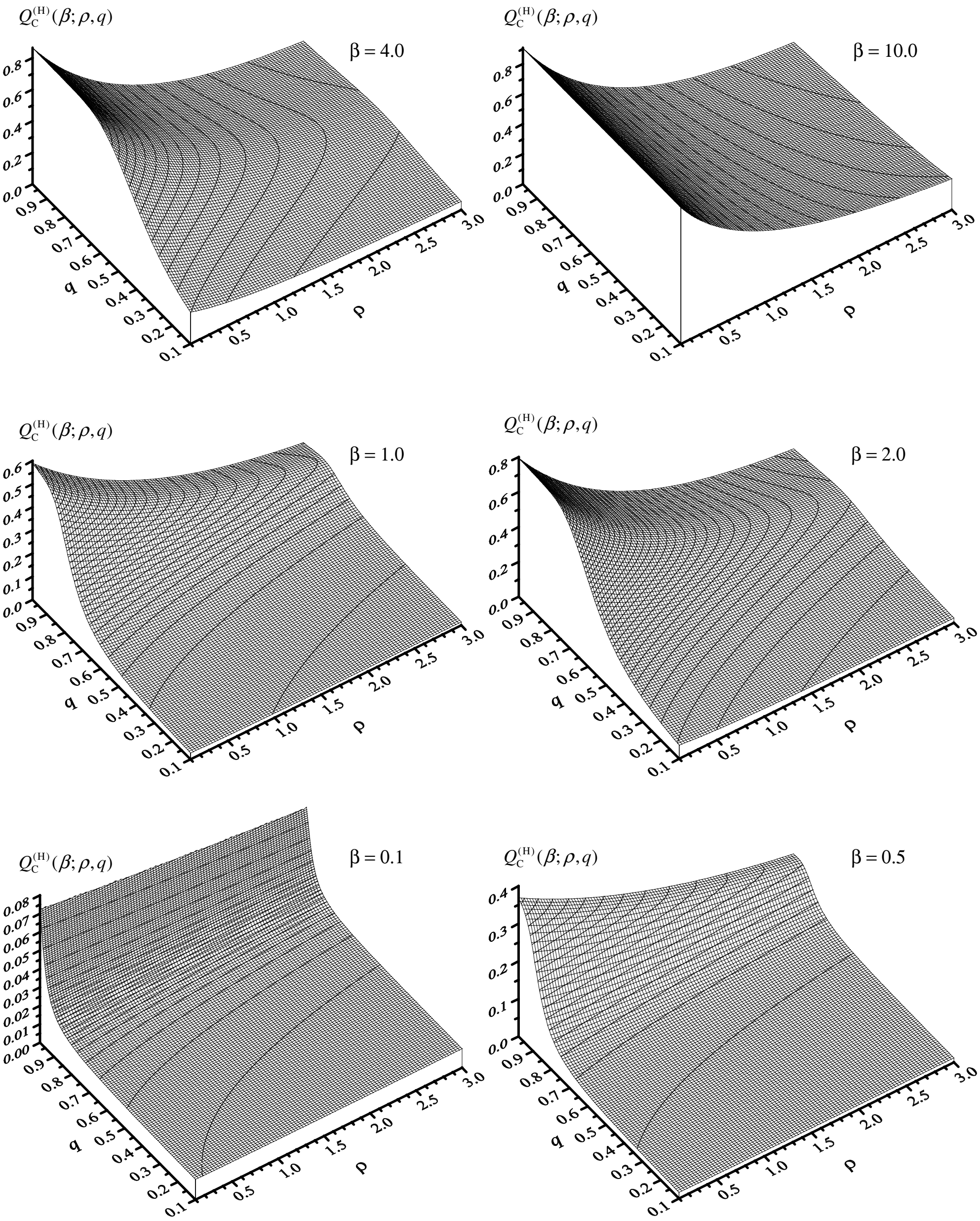

Figure 2. The set of six stages showing the thermal evolution of the Husimi function $Q_C^{(H)}$ in terms of the quantum state expansion factor ρ and the temperature coefficient β, evaluated for a purely coherent state $|z; a_j\rangle_C$. To help the visualization of the $Q_C^{(H)}$-surface behaviour, we set in each small figure some level curves.

non-orthogonal. In addition, we observe that the diagonal representation of the density operator in purely coherent states (the Husimi's Q-function), obtained in equation (46), is convenient for evaluating expectations of the different operators concerning the self-similar potential system. In this sense we perform the diagonal expansion of the density operator in the purely coherent states basis:

$$\hat{\varrho}_\beta^{(\mathrm{C})} = \frac{1}{Z_\beta}\int \mathrm{d}^2 z\, w_{\mathrm{C}}(\rho_{\mathrm{o}}, q)|z; a_j\rangle_{\mathrm{C}} P_{\mathrm{C}}(\beta; \rho_{\mathrm{o}}, q)_{\mathrm{C}}\langle z; a_j| \tag{50}$$

where the conditions about the distribution function $P_{\mathrm{C}}(\beta; \rho_{\mathrm{o}}, q)$ can be obtained from the diagonal expansion $\hat{\varrho}_\beta^{(\mathrm{C})}$ of the density operator and the relation $\langle\Psi|\hat{\varrho}_\beta|\Phi\rangle = \langle\Psi|\hat{\varrho}_\beta^{(\mathrm{C})}|\Phi\rangle$ which must be fulfilled for any arbitrary states $\langle\Psi|$ and $|\Phi\rangle$ from the Hilbert space (or for any states from the basis $|z; a_j\rangle_{\mathrm{C}}$ or $|n\rangle$). Using equations (40) and (50) in this relation we can prove that the following condition must be satisfied,

$$\int_0^\infty \mathrm{d}\rho_{\mathrm{o}}\, \rho_{\mathrm{o}}^n \mathcal{N}_{\mathrm{C}}^2(\rho_{\mathrm{o}}, a_j) w_{\mathrm{C}}(\rho_{\mathrm{o}}, q) P_{\mathrm{C}}(\beta; \rho_{\mathrm{o}}, q) = \mathrm{e}^{-\beta e_n}\left|h_n^{(\mathrm{C})}(a_j)\right|^2, \tag{51}$$

where the normalization function $\mathcal{N}_{\mathrm{C}}(\rho_{\mathrm{o}}, a_j)$, the weight function $w_{\mathrm{C}}(\rho_{\mathrm{o}}, q)$ and the expansion coefficient $h_n^{(\mathrm{C})}(a_j)$ are given by equations (35), (48) and (34), respectively. Since we are interested in obtaining the thermal expectation value of the observables related to the correlation functions for the self-similar potential system, we observe that it is sufficient to use relation (51) obtained above. Indeed, we can evaluate the thermal expectation value of $\mathcal{H}_-^k$ through of the expression

$$\langle\hat{\mathcal{H}}_-^k(\beta, q)\rangle_{\mathrm{C}} = \mathrm{Tr}\left\{\hat{\varrho}_\beta^{(\mathrm{C})}\hat{\mathcal{H}}_-^k\right\} = \frac{1}{Z_\beta}\int \mathrm{d}^2 z\, w_{\mathrm{C}}(\rho_{\mathrm{o}}, q) P_{\mathrm{C}}(\beta; \rho_{\mathrm{o}}, q)_{\mathrm{C}}\langle z; a_j|\hat{\mathcal{H}}_-^k|z; a_j\rangle_{\mathrm{C}} \tag{52}$$

and, when we consider the expectation value (42) in this expression, we find that

$$\langle\hat{\mathcal{H}}_-^k(\beta, q)\rangle_{\mathrm{C}} = \frac{1}{Z_\beta}\sum_{n=0}^\infty \frac{e_n^k}{\left|h_n^{(\mathrm{C})}(a_j)\right|^2}\int_0^\infty \mathrm{d}\rho_{\mathrm{o}}\, \rho_{\mathrm{o}}^n \mathcal{N}_{\mathrm{C}}^2(\rho_{\mathrm{o}}, a_j) w_{\mathrm{C}}(\rho_{\mathrm{o}}, q) P_{\mathrm{C}}(\beta; \rho_{\mathrm{o}}, q). \tag{53}$$

Therefore, with (51) into (53) it is possible to express the thermal expectation value of $\mathcal{H}_-^k$ through the β-derivatives of the Z_β:

$$\langle\hat{\mathcal{H}}_-^k(\beta, q)\rangle_{\mathrm{C}} = \frac{1}{Z_\beta}\sum_{n=0}^\infty e_n^k\, \mathrm{e}^{-\beta e_n} = \frac{(-1)^k}{Z_\beta}\frac{\partial^k Z_\beta}{\partial\beta^k}. \tag{54}$$

In this manner the thermal expectations for the first two powers are

$$\langle\hat{\mathcal{H}}_-(\beta, q)\rangle_{\mathrm{C}} = -\frac{\partial \ln Z_\beta}{\partial\beta} \qquad \text{and} \qquad \langle\hat{\mathcal{H}}_-^2(\beta)\rangle_{\mathrm{C}} = \frac{\partial^2 \ln Z_\beta}{\partial\beta^2} + \left(\frac{\partial \ln Z_\beta}{\partial\beta}\right)^2. \tag{55}$$

In a similar way, it is possible to express all thermodynamical and statistical quantities characteristics of the canonical self-similar potential ensemble as functions of the $\ln Z_\beta$. For example, the internal energy $U_{\mathrm{C}}(\beta, q)$ and the heat capacity $C_{\mathrm{C}}(\beta, q)$ of the system may be obtained through

$$U_{\mathrm{C}}(\beta, q) = \langle\hat{\mathcal{H}}_-(\beta, q)\rangle_{\mathrm{C}} \qquad \text{and} \qquad C_{\mathrm{C}}(\beta, q) = \frac{\partial U_{\mathrm{C}}(\beta, q)}{\partial T} = \beta^2\frac{\partial^2 \ln Z_\beta}{\partial\beta^2}. \tag{56}$$

On the other hand, taking into account the results of equations (55), we can show that the thermal analogue for the Mandel parameter $Q_{\mathrm{C}}^{(\mathrm{M})}(\beta, q)$ is obtained by

$$Q_{\mathrm{C}}^{(\mathrm{M})}(\beta, q) = 1 - \frac{\partial^2 \ln Z_\beta}{\partial\alpha_{\mathrm{o}}^2}\bigg/\left(\frac{\partial \ln Z_\beta}{\partial\alpha_{\mathrm{o}}}\right) \qquad \text{where} \qquad \alpha_{\mathrm{o}} = \beta R(a_1). \tag{57}$$

Before we conclude this section, we note that in the harmonic oscillator limit $q \to 1$ we have $e_n \to nR(a_1)$ and the partition function goes to $Z_\beta \to 1/\{1 - \mathrm{e}^{-\alpha_{\mathrm{o}}}\}$ while the first expectation values of $\hat{\mathcal{H}}_-^k$ go to the limit expressions given by $\langle\hat{\mathcal{H}}_-(\beta, q)\rangle_{\mathrm{C}} = U_{\mathrm{C}}(\beta, q) \to$

$R(a_1)/\{e^{\alpha_o} - 1\}$ and $\langle \hat{\mathcal{H}}_-^2(\beta, q)\rangle_C \to R^2(a_1)\{e^{\alpha_o} + 1\}/\{e^{\alpha_o} - 1\}^2$. In this case we find the heat capacity for the harmonic oscillator model of a perfect gas $C_C(\beta, q) \to R^2(a_1)\, e^{\alpha_o}/(e^{\alpha_o} - 1)^2$ and the thermal evaluation of the Mandel parameter giving the characteristic result for the harmonic oscillator potential system $Q_C^{(M)}(\beta, q) \to 1/(e^{\alpha_o} - 1)$.

4.2. Thermal effects via purely squeezed states

In the same way, since $\langle n|\hat{\mathcal{H}}_-^k|n'\rangle = e_n^k \delta_{nn'}$, we find for the expectation value of $\hat{\mathcal{H}}_-^k$ in the purely squeezed state $|z; a_j\rangle_S$ for the self-similar potential system

$$\langle \hat{\mathcal{H}}_-^k(\rho, q)\rangle_S \equiv {}_S\langle z; a_j|\hat{\mathcal{H}}_-^k|z; a_j\rangle_S = \frac{1}{(-q^2\rho; q^2)_\infty} \sum_{n=0}^{\infty} \frac{e_{2n}^k q^{n(n+1)} \rho^n}{(q^2; q^2)_n} \tag{58}$$

where we used relations (19) and (20). Taking into account relation (9) and the definition of the q-shifted factorial $(q; q)_n$, after some calculations we can prove that

$$\sum_{n=0}^{\infty} \frac{e_{2n}^k q^{n(n+1)} \rho^n}{(q^2; q^2)_n} = \left[\frac{R(a_1)}{1 - q}\right]^k \sum_{m=0}^{k} (-1)^m \binom{k}{m} (-\rho q^{2m+2}; q^2)_\infty \tag{59}$$

and using this result in (58) we find an expression for $\langle \hat{\mathcal{H}}_-^k(\rho, q)\rangle_S$ in the same form obtained in (44) for the purely coherent case of the previous section with the ratio parcel factor $\chi_q^{(-m)}$ replaced by

$$\zeta_q^{(m)} \equiv \frac{(-\rho q^{2m+2}; q^2)_\infty}{(-\rho q^2; q^2)_\infty}. \tag{60}$$

Using this general result we can show that the Mandel parameter for the self-similar potential system in a purely squeezed state, defined by the equivalent expression $Q_S^{(M)} \equiv \langle \hat{h}_-\rangle_S \big\{\big[\langle \hat{h}_-^2\rangle_S - \langle \hat{h}_-\rangle_S\big]\big/\langle \hat{h}_-\rangle_S^2 - 1\big\}$, is obtained by using an expression with the same form presented by equation (45), with the m-order correspondent ratio factor $\chi_q^{(-m)}$ replaced by the $\zeta_q^{(m)}$.

We show in figure 3 the contour plots of the Mandel parameter surface on the $(q\rho)$-plane for the purely coherent state case (top part of the figure) and the purely squeezed state case (bottom part of the figure). To help the visualization, we present each level in the figure with a different shading (darker regions are related to lower observable values). Comparing the results shown in both cases we observe the resemblance of behaviour between the Mandel parameters $Q_C^{(M)}(\rho, q)$ and $Q_S^{(M)}(\rho, q)$. From the figure we observe: (i) the super-Poissonian statistic nature of the purely coherent state $|z; a_j\rangle_C$ and of the purely squeezed state $|z; a_j\rangle_S$ for the self-similar potential system; (ii) the low sensitivity of the Mandel parameters $Q_C^{(M)}(\rho, q)$ and $Q_S^{(M)}(\rho, q)$ in relation to the squeezing parameter q; (iii) the higher values assumed by $Q_C^{(M)}(\rho, q)$ in relation to $Q_S^{(M)}(\rho, q)$ inside the $(q\rho)$-plane limits presented in the figure.

On the other hand, using relations (19) and (20) we can evaluate the Husimi's Q-function in a purely squeezed state representation, in this case defined as $Q_S^{(H)}(\beta; \rho_o, q) \equiv {}_S\langle z; a_j|\hat{\varrho}_\beta|z; a_j\rangle_S$, and obtain the result

$$Q_S^{(H)}(\beta; \rho_o, q) = \frac{\mathcal{N}_S^2(\rho_o, a_j)}{Z_\beta} \sum_{n=0}^{\infty} \frac{e^{-\beta e_{2n}} \rho_o^n}{\big|h_n^{(S)}(a_j)\big|^2} = \frac{1}{Z_\beta(-q^2\rho; q^2)_\infty} \sum_{n=0}^{\infty} \frac{e^{-\beta e_{2n}} q^{n(n+1)} \rho^n}{(q^2; q^2)_n}. \tag{61}$$

Figure 4 is the purely squeezed state version of figure 2, showing, in a set of six stages, the thermal evolution of the Husimi $Q_S^{(H)}$-surface as a function of the quantum state expansion

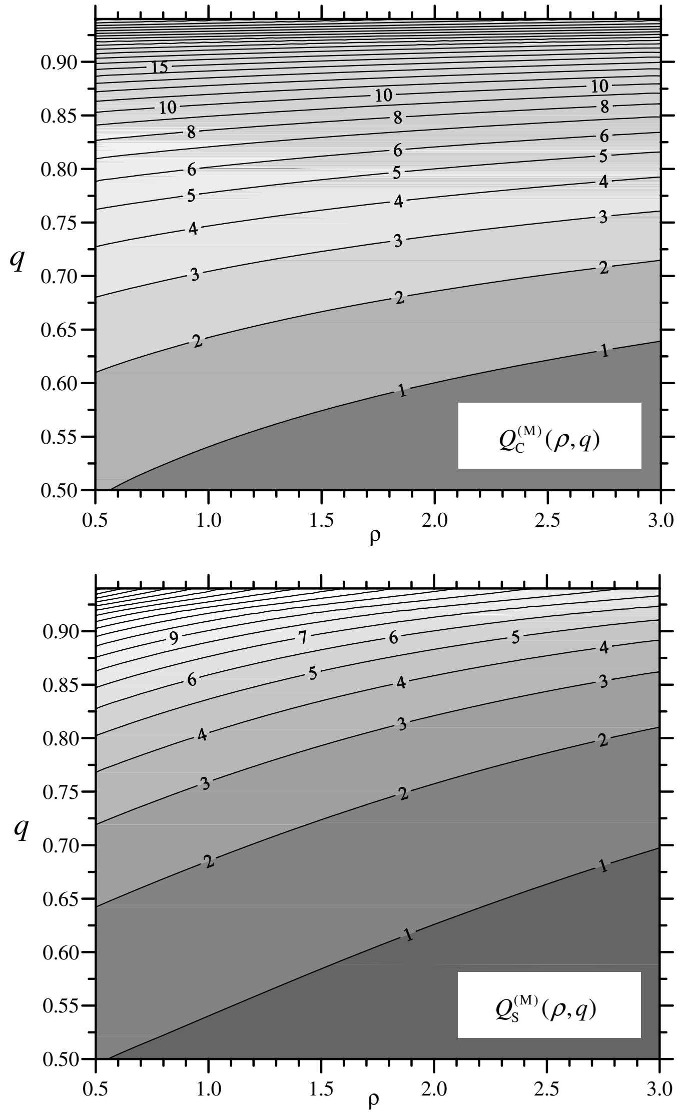

Figure 3. Contour plots of the Mandel parameters $Q_{\mathrm{C}}^{(\mathrm{M})}(\rho, q)$ and $Q_{\mathrm{S}}^{(\mathrm{M})}(\rho, q)$ on the $(q\rho)$-plane, evaluated for a purely coherent state $|z; a_j\rangle_{\mathrm{C}}$ and for a purely squeezed state $|z; a_j\rangle_{\mathrm{S}}$, respectively.

factor ρ and the temperature coefficient β. As in the purely coherent state case, to help the visualization of the $Q_{\mathrm{S}}^{(\mathrm{H})}$-surface behaviour, we set in each small figure some of its constant level curves. Following the sequence of small figures with the increase in the temperature coefficient β, we observe: (i) the increase of the $Q_{\mathrm{S}}^{(\mathrm{H})}(\beta; \rho, q)$ values according to the increase in the β values; (ii) the appearance of a maximum folding in the $Q_{\mathrm{S}}^{(\mathrm{H})}$-surface, the height and width of which increase with β while its position moves to lower values of q. This maximum folding vanishes for higher values of β; (iii) that there is some resemblance between the behaviour of $Q_{\mathrm{C}}^{(\mathrm{H})}(\beta; \rho, q)$ shown in figure 3 and the behaviour of $Q_{\mathrm{S}}^{(\mathrm{H})}(\beta; \rho, q)$ shown in this figure for intermediate values of β, but not for the lowest and the highest values of β used in the calculation for these figures.

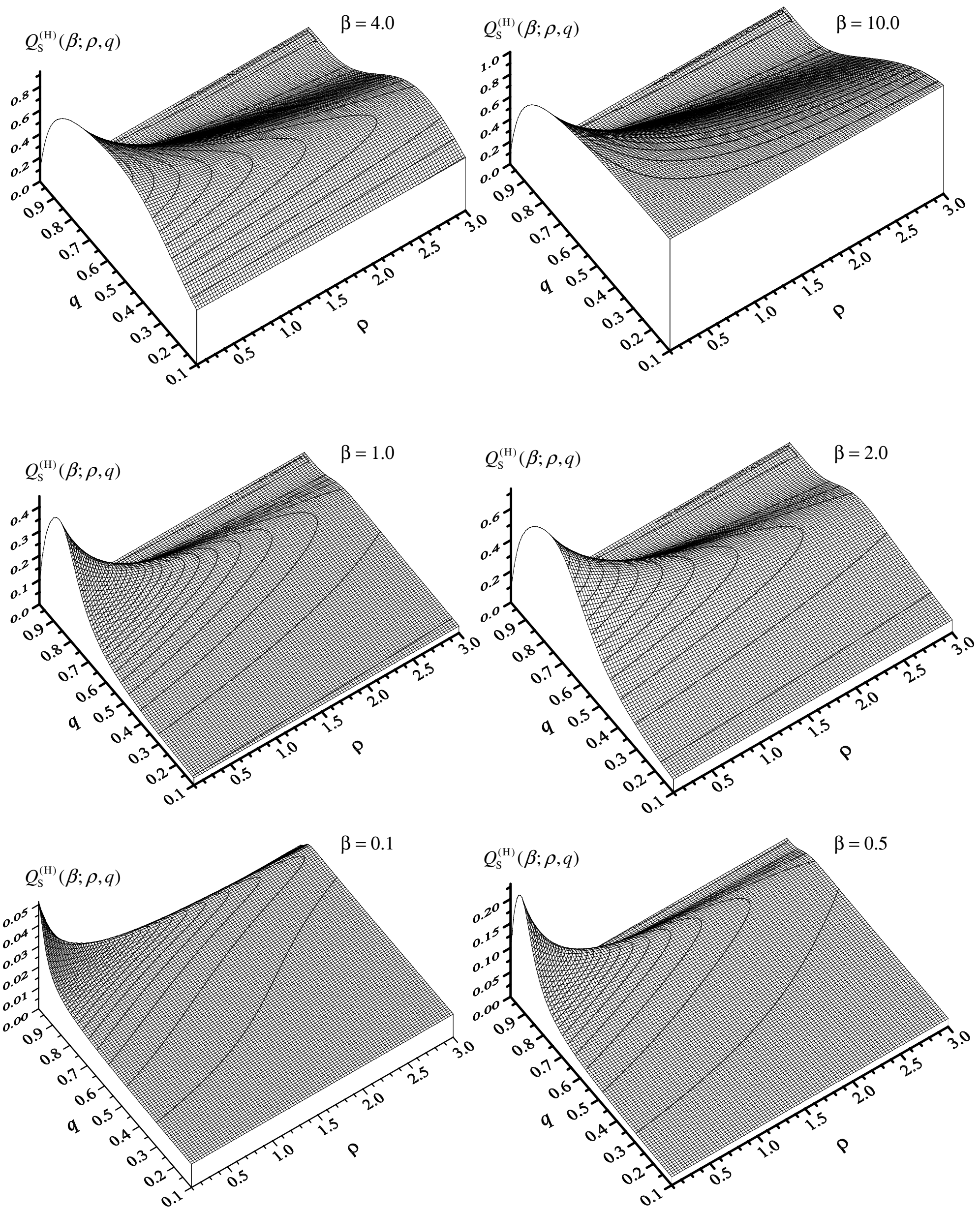

Figure 4. Same as figure 2 calculated for a purely squeezed state $|z; a_j\rangle_S$.

Looking at the purely squeezed state Glauber expansion (20) we observe that the representation of any observable in such kind of states must be restricted to the subspace $\mathcal{E}_+$ of the states with even quantum numbers $2n$ of the Hilbert space $\mathcal{E} = \mathcal{E}_+ + \mathcal{E}_-$. Under these conditions the following partial normalization relation must be satisfied:

$$\mathrm{Tr}_S\, \hat{\varrho}_\beta = \int \mathrm{d}^2 z\, w_S(\rho_o, q) Q_S^{(H)}(\beta; \rho_o, q) = \frac{Z_\beta^{(+)}}{Z_\beta} \tag{62}$$

with the partition function decomposed in the even and odd parts $Z_\beta = Z_\beta^{(+)} + Z_\beta^{(-)}$, where $Z_\beta^{(+)} = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta e_{2n}}$ and $Z_\beta^{(-)} = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta e_{2n+1}}$. It is straightforward to verify that condition (62) is satisfied when we assume the existence of the positive-definite weight function

$$w_{\mathrm{S}}(\rho_{\mathrm{o}}, q) = \frac{(-q^2\rho; q^2)_\infty}{\pi \log(1/q^2)(-\rho; q^2)_\infty} \tag{63}$$

and we use the Ramanujan's integral (49) with the scaling parameter q replaced by q^2.

Since the purely squeezed states $|z; a_k\rangle_{\mathrm{S}}$ for a self-similar potential system form an overcomplete set of states in the Hilbert subspace $\mathcal{E}_+$, they can be used as a basis set in this subspace despite the fact that they are non-orthogonal. Under these conditions, we perform the diagonal expansion of the density operator in the purely squeezed states basis:

$$\hat{\varrho}_\beta^{(\mathrm{S})} = \left\{ \frac{1}{Z_\beta} \int \mathrm{d}^2 z\, w_{\mathrm{S}}(\rho_{\mathrm{o}}, q) |z; a_j\rangle_{\mathrm{S}} P_{\mathrm{S}}(\beta; \rho_{\mathrm{o}}, q)_{\mathrm{S}}\langle z; a_j| \right\} \Big/ \left(\frac{Z_\beta^{(+)}}{Z_\beta} \right) \tag{64}$$

where the conditions about the distribution function $P_{\mathrm{S}}(\beta; \rho_{\mathrm{o}}, q)$ can be obtained from the diagonal expansion of the density operator $\hat{\varrho}_\beta^{(\mathrm{S})}$, observing that the relation $\langle\Psi|\hat{\varrho}_\beta|\Phi\rangle = \langle\Psi|\hat{\varrho}_\beta^{(\mathrm{S})}|\Phi\rangle$ must be fulfilled for any arbitrary states $\langle\Psi|$ and $|\Phi\rangle$ from the Hilbert subspace $\mathcal{E}_+$ (or for any states from the basis $|z; a_j\rangle_{\mathrm{S}}$ or $|2n\rangle$). Taking into account equations (40) and (64), we can prove that the following condition must be satisfied:

$$\int_0^\infty \mathrm{d}\rho_{\mathrm{o}}\, \rho_{\mathrm{o}}^n \mathcal{N}_{\mathrm{S}}^2(\rho_{\mathrm{o}}, a_j) w_{\mathrm{S}}(\rho_{\mathrm{o}}, q) P_{\mathrm{S}}(\beta; \rho_{\mathrm{o}}, q) = \mathrm{e}^{-\beta e_{2n}} \left| h_n^{(\mathrm{S})}(a_j) \right|^2, \tag{65}$$

where the normalization function $\mathcal{N}_{\mathrm{S}}(\rho_{\mathrm{o}}, a_j)$, the weight function $w_{\mathrm{S}}(\rho_{\mathrm{o}}, q)$ and the expansion coefficient $h_n^{(\mathrm{S})}(a_j)$ are given by equations (20), (63) and (19), respectively.

Since we are interested in obtaining the thermal expectation value of the observables related to the correlation functions for the self-similar potential system, we observe that it is enough for us to use relation (65) obtained above. Indeed, we can evaluate the expectation value

$$\langle \hat{\mathcal{H}}_-^k(\beta, q)\rangle_{\mathrm{S}} = \mathrm{Tr}\left\{ \hat{\varrho}_\beta^{(\mathrm{S})} \hat{\mathcal{H}}_-^k \right\} = \frac{1}{Z_\beta^{(+)}} \int \mathrm{d}^2 z\, w_{\mathrm{S}}(\rho_{\mathrm{o}}, q) P_{\mathrm{S}}(\beta; \rho_{\mathrm{o}}, q)_{\mathrm{S}}\langle z; a_j|\hat{\mathcal{H}}_-^k|z; a_j\rangle_{\mathrm{S}} \tag{66}$$

and, when we use that ${}_{\mathrm{S}}\langle z; a_j|\hat{\mathcal{H}}_-^k|z; a_j\rangle_{\mathrm{S}} = \mathcal{N}_{\mathrm{S}}^2(\rho_{\mathrm{o}}, a_j) \sum_{n=0}^{\infty} e_{2n}^k \rho_{\mathrm{o}}^n \big/ \left| h_n^{(\mathrm{S})}(a_j) \right|^2$ we find that

$$\langle \hat{\mathcal{H}}_-^k(\beta, q)\rangle_{\mathrm{S}} = \frac{1}{Z_\beta^{(+)}} \sum_{n=0}^{\infty} \frac{e_{2n}^k}{\left| h_n^{(\mathrm{S})}(a_j) \right|^2} \int_0^\infty \mathrm{d}\rho_{\mathrm{o}}\, \rho_{\mathrm{o}}^n \mathcal{N}_{\mathrm{S}}^2(\rho_{\mathrm{o}}, a_j) w_{\mathrm{S}}(\rho_{\mathrm{o}}, q) P_{\mathrm{S}}(\beta; \rho_{\mathrm{o}}, q). \tag{67}$$

Therefore, with (65) into (67) we can prove that

$$\langle \hat{\mathcal{H}}_-^k(\beta, q)\rangle_{\mathrm{S}} = \frac{1}{Z_\beta^{(+)}} \sum_{n=0}^{\infty} e_{2n}^k \, \mathrm{e}^{-\beta e_{2n}} = \frac{(-1)^k}{Z_\beta^{(+)}} \frac{\partial^k Z_\beta^{(+)}}{\partial \beta^k}. \tag{68}$$

Taking into account this general result, we find that the internal energy $U_{\mathrm{S}}(\beta, q)$, the heat capacity $C_{\mathrm{S}}(\beta, q)$ and the thermal Mandel parameter $Q_{\mathrm{S}}^{(\mathrm{M})}(\beta, q)$ in the purely squeezed states representation are obtained by expressions with the same form presented, respectively, in equations (56) and (57) of the previous section with the partition function Z_β replaced by its even part $Z_\beta^{(+)}$. The following three figures show the behaviour of these thermal quantities in terms of the temperature coefficient β and the scaling parameter q and compare the results obtained for the purely coherent state case with the purely squeezed state case.

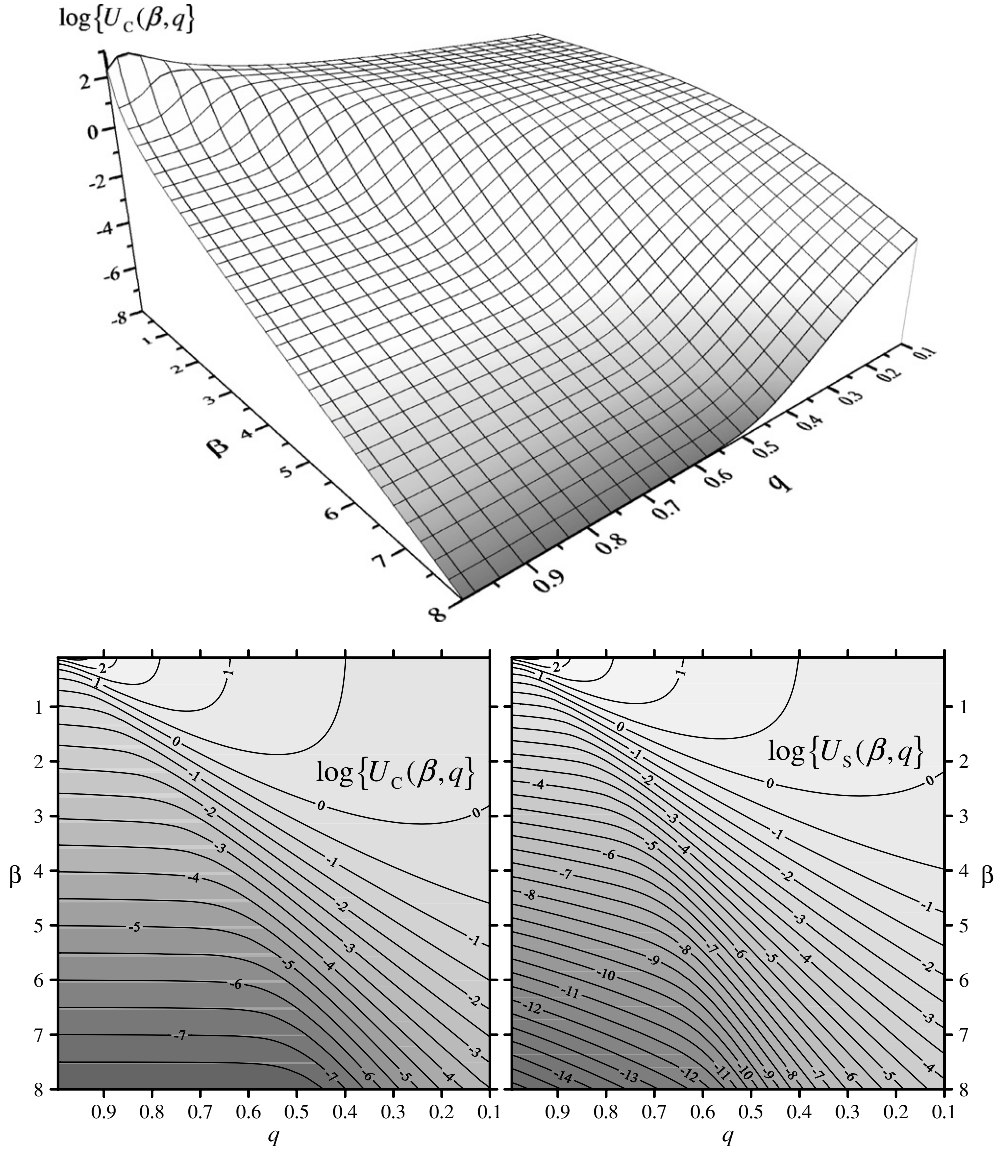

Figure 5. In the top part we have the three-dimensional plot of the surface of $\log\{U_C(\beta, q)\}$ as a function of β and q for the purely coherent state case. Bottom part shows the contour plots of $\log\{U_C(\beta, q)\}$ and $\log\{U_S(\beta, q)\}$ as a function of β and q for the purely coherent state and for the purely squeezed states cases, respectively.

Figure 5 displays in its top part the three-dimensional plot of the surface of the logarithm of the internal energy $U_C(\beta, q)$ as a function of β and q for the purely coherent state case. To make the observation of the differences between the behaviour of $U_C(\beta, q)$ and the behaviour of the internal energy $U_S(\beta, q)$, obtained in the purely squeezed state case, easier we show in the bottom part of the figure the contour plots of the logarithm of the internal energy surface on the (β, q)-plane for both the cases. To help the visualization, we present each level in the figure with a different shading (darker regions are related to lower observable values).

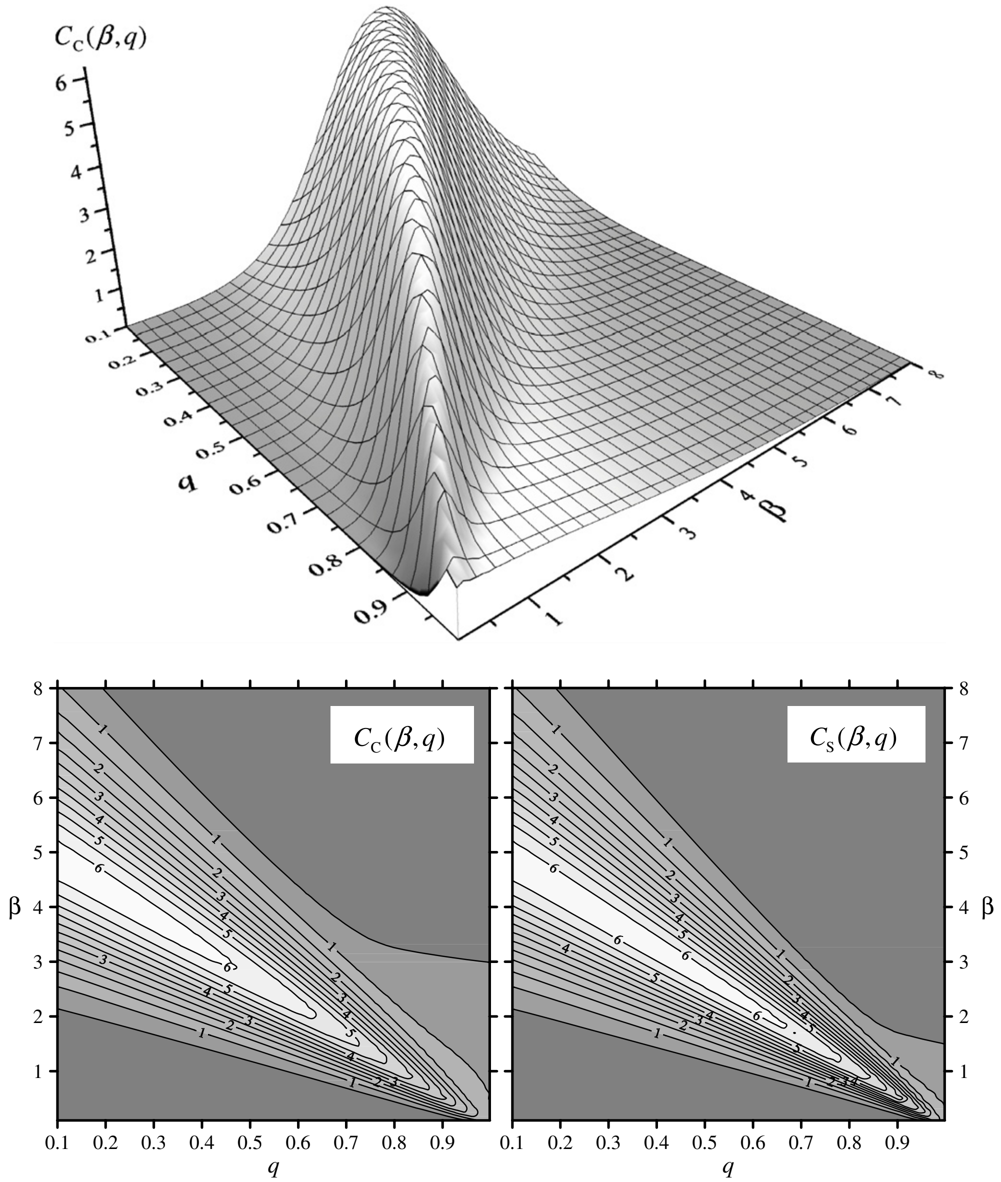

Figure 6. In the top part we have the three-dimensional plot of the surface of $C_C(\beta, q)$ as a function of β and q for the purely coherent state case. Bottom part shows the contour plots of $C_C(\beta, q)$ and $C_S(\beta, q)$ as a function of β and q for the purely coherent state and for the purely squeezed states cases, respectively.

From the figure we observe: (i) the almost linear increasing of $U_C(\beta, q)$ with β and the low sensibility with q in the region of the plane (β, q) below the straight line $\beta(q) = 16(1 - q)$; (ii) in this region $U_S(\beta, q)$ shows lower values than $U_C(\beta, q)$ and some sensibility with q; (iii) beyond the diagonal linking the corners (0, 1) and (8, 0.1) both of internal energies $U_C(\beta, q)$ and $U_S(\beta, q)$ show increasing behaviour with the greatest values concentrated around the corner (0, 1); (iv) the behaviour of the two internal energies is very similar in this region,

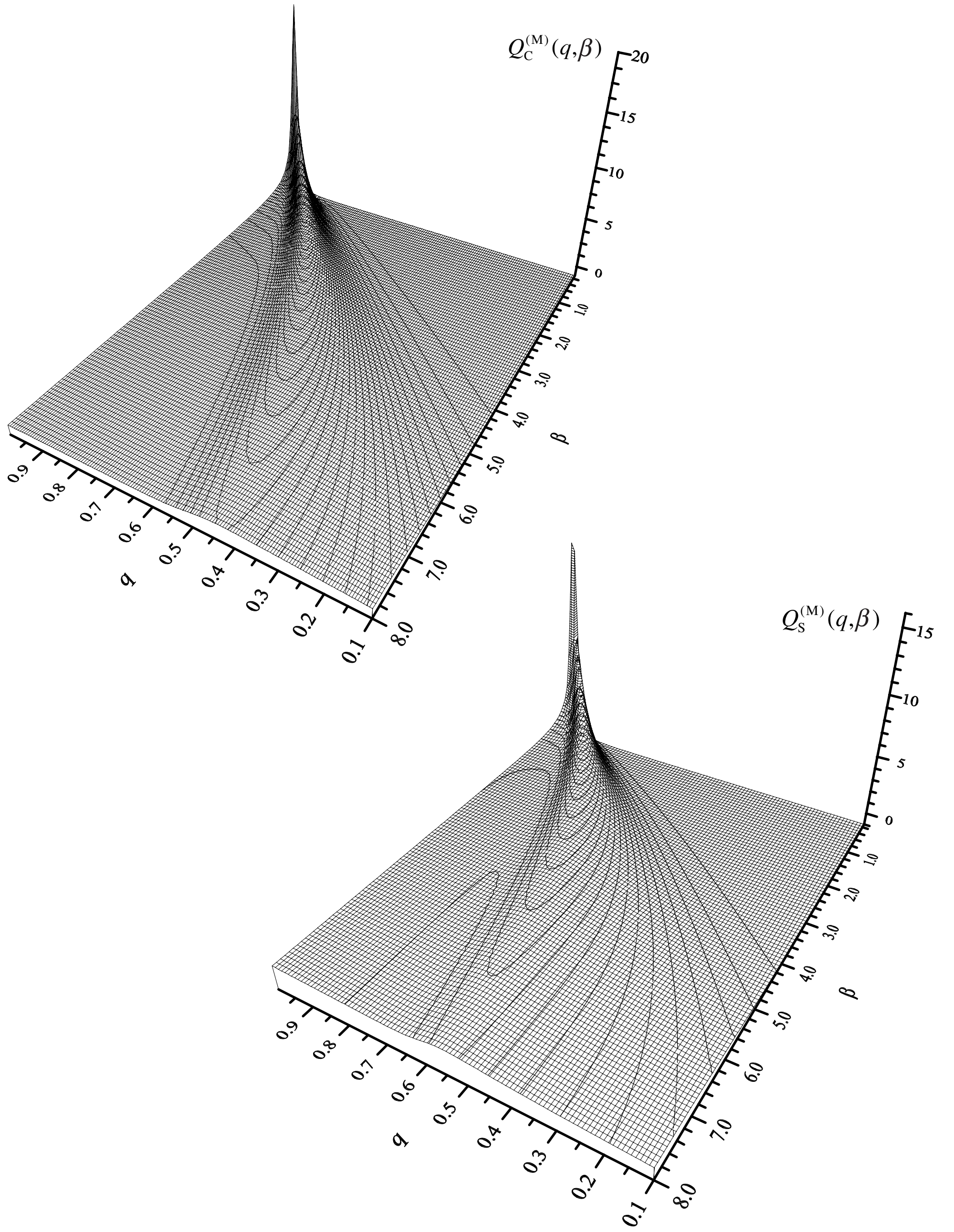

Figure 7. The three-dimensional plots, in terms of β and q, of the thermal Mandel parameters $Q_{\mathrm{C}}^{(\mathrm{M})}(\beta, q)$ and $Q_{\mathrm{S}}^{(\mathrm{M})}(\beta, q)$ evaluated for a purely coherent state $|z; a_j\rangle_{\mathrm{C}}$ and for a purely squeezed state $|z; a_j\rangle_{\mathrm{S}}$, respectively.

however we observe that $U_{\mathrm{S}}(\beta, q)$ shows a more abrupt transition and a bigger concentration next to the $\beta = 0.1$ side of the figure.

Figure 6 is the version of figure 5 for the heat capacity case, showing in its top part the three-dimensional plot of the $C_{\mathrm{C}}(\beta, q)$-surface as a function of β and q. From the figure we observe: (i) the appearing of a folding in the heat capacity surface localized along the straight line $\beta(q) = 5(1 - q)$ where $C_{\rm C}(\beta, q)$ shows great values; (ii) this folding increases its height and width from the corner (0.1, 1) to the side $q = 0.1$ of the figure; (iii) the behaviour of $C_{\rm S}(\beta, q)$ is very similar to $C_{\rm C}(\beta, q)$, however, like the internal energy case, shows a more abrupt transition in the folding borders.

In figure 7 we show three-dimensional plots of the surfaces of the thermal Mandel parameter as a function of the self-similar scaling parameter q and the temperature coefficient β for the purely coherent state case (top part of the figure) and the purely squeezed state case (bottom part of the figure). To help the visualization, we set in this figure some level curves of constant $Q_{\rm C}^{(\rm M)}(\beta, q)$ and $Q_{\rm S}^{(\rm M)}(\beta, q)$ values. We observe from the figure the resemblance between the behaviour of the thermal Mandel parameter in the two cases. In both of them we observe the appearing of a very accentuated folding in the surface (such as a jaws hull) situated around the diagonal of the $(q\beta)$-plane, the height of which increases when $\beta \to 0$ and $q \to 1$. Note from the $Q_{\rm C}^{(\rm M)}(\beta, q)$ and $Q_{\rm S}^{(\rm M)}(\beta, q)$ behaviour that the purely coherent state $|z; a_j\rangle_{\rm C}$ and the purely squeezed state $|z; a_j\rangle_{\rm S}$ change from a sub-Poissonian statistic (in the bottom plane of the thermal Mandel parameter) to a super-Poissonian statistic (in the surface folding extension of the thermal Mandel parameter).

5. Final remarks

In this paper, using an algebraic approach, we studied squeezing and thermal effects for self-similar potential systems. We introduced quadrature operators $(\hat{X}, \hat{P})$ and evaluated the behaviour, in terms of the scaling parameter q and of the quantum state expansion factor ρ, of the squeezing resulting from two possible quantum states of the systems: a purely squeezed state and a composite squeezed state resulting from the superposition of two purely coherent states.

Taking into account that the self-similar potential system can be associated with an anharmonic potential system, where the anharmonic deviations are related to the scaling parameter q, we studied the statistical properties, in terms of q and of the Boltzmann's temperature coefficient β, of a canonical ensemble constituted by self-similar potential systems. We evaluated the Husimi's function $Q^{(\rm H)}(\beta; \rho, q)$, the internal energy $U(\beta, q)$, the heat capacity $C(\beta, q)$ and the thermal Mandel parameter $Q^{(\rm M)}(\beta, q)$ when the density operator ϱ_β is expanded in purely coherent states $|z; a_j\rangle_{\rm C}$ as well as in purely squeezed states $|z; a_j\rangle_{\rm S}$ and compared the results obtained in the two representations.

Acknowledgments

This work was supported in part by the US National Science Foundation grant no. PHY-0555231 at the University of Wisconsin and in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation.

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